

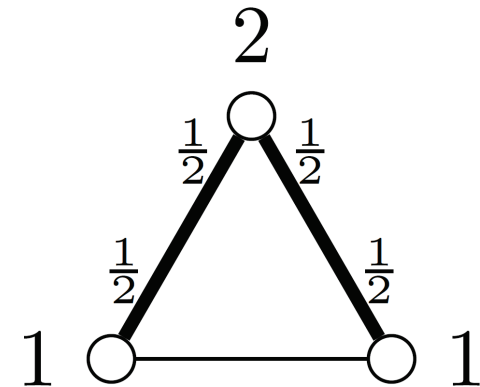
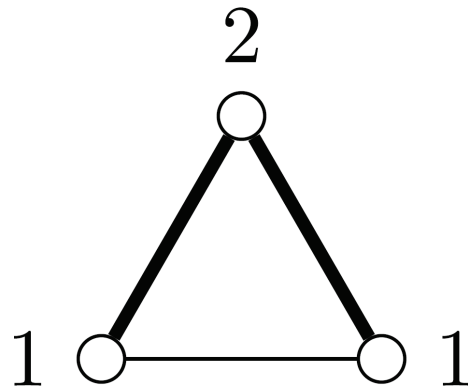
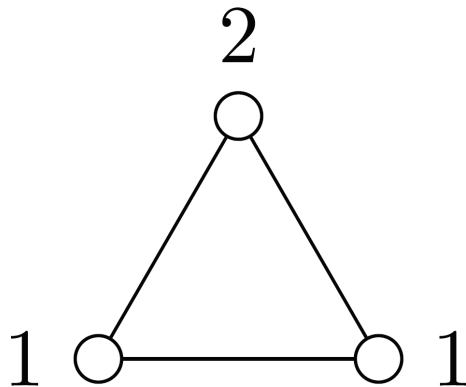
Capacitated Network Bargaining Games: Stability and Structure

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Based on joint work with Laura Sanità

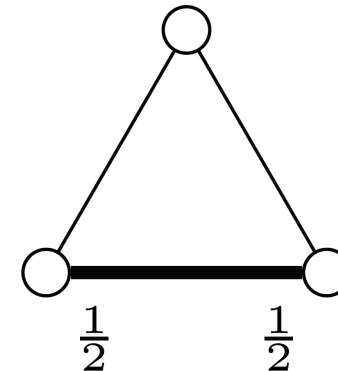
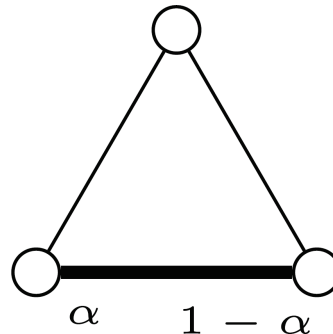
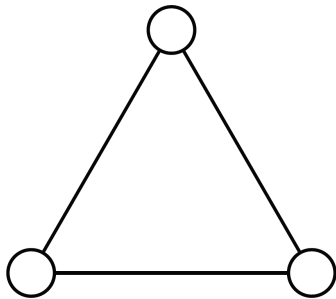
Capacitated Network Bargaining Games

- Given: graph $G = (V, E)$, edge weights $w \in \mathbb{R}_{\geq 0}^E$, vertex capacities $c \in \mathbb{Z}_{\geq 0}^V$.
- V : players
- E : potential deals with value w
- A player v can enter in c_v deals: a set of deals is a c-matching.
- Players decide how to split the value of their deal:
$$a_{uv} + a_{vu} = w_{uv} \text{ if } uv \in M \text{ and } a_{uv} = a_{vu} = 0 \text{ otherwise.}$$
- An outcome is *stable* if no pair of players has an incentive to break the current set of deals to enter in a deal with each other.



Stability

- Not in all graphs a stable outcome is possible.
→ Modify the graph so that a stable outcome does exist, we call this *stabilizing*.
- We consider two modifications:
 - capacity reduction,
 - edge removal.
- In unit-capacity ($c = 1$) graphs these problems are well studied:
 - Capacity reduction (or vertex removal): polynomial-time solvable [AHS18, IK+, KS20].
 - Edge removal: hard-to-approximate with a constant factor, but there is an $O(\max \text{ degree})$ -approximation algorithm [BC+15, KS20].



Structure

- [BH+10] There exists a stable outcome for the game defined on G if and only if there exists an integral optimal solution to the following LP

$$\max\{w^T x : x(\delta(v)) \leq c_v \ \forall v \in V, 0 \leq x \leq 1\}.$$

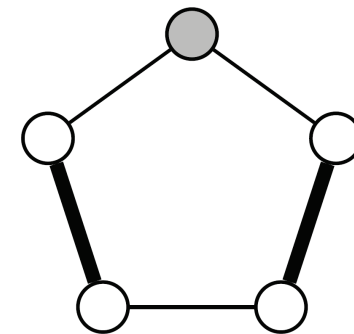
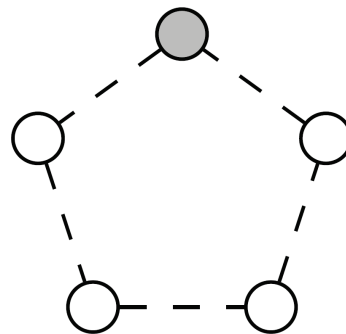
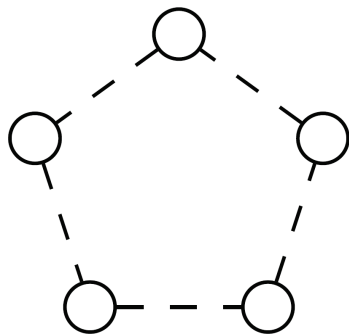
- We call feasible solutions of this LP *fractional c-matchings*.
- Extreme point solutions of this LP satisfy $x_e \in \{0, \frac{1}{2}, 1\}$ for all edges $e \in E$, and the edges with $x_e = \frac{1}{2}$ induce vertex-disjoint odd cycles with saturated vertices.

→ We want to get rid of these odd cycles.

Capacity Reduction

Theorem. Stabilizing by reducing capacity is polynomial-time solvable.

- Idea of the algorithm:
 - Compute a basic maximum-weight fractional c-matching with minimum number of odd cycles induced by the $x_e = \frac{1}{2}$ edges.
 - For each such odd cycle, choose one vertex and reduce its capacity by one.
- This works because the minimum number of these odd cycles is a lower bound on the capacity that has to be reduced. ★



Capacity Reduction

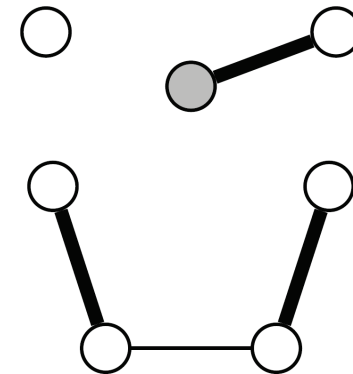
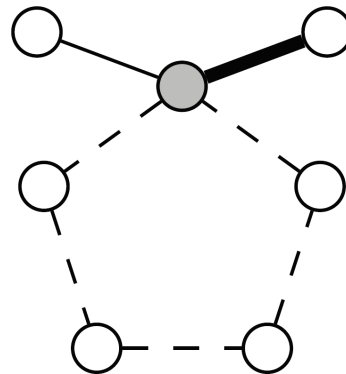
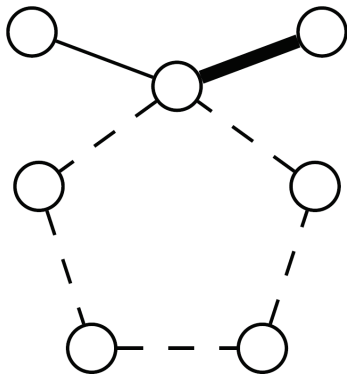
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- Idea of the algorithm:
 - Compute a basic maximum-weight fractional c-matching with minimum number of odd cycles induced by the $x_e = \frac{1}{2}$ edges.
 - For each such odd cycle, choose one vertex and reduce its capacity by one.
- Key properties of our algorithm:
 - It preserves the total value that the players can get up to a factor that is asymptotically best possible.
 - It reduces the capacity of each player by at most one.

Edge Removal

Theorem. There is an $O(\text{max degree})$ -approximation algorithm for stabilizing by removing edges.

- Idea of the algorithm:
 - Use the capacity reduction algorithm.
 - For each vertex that would have its capacity reduced, remove all edges incident with it, except the ones with $x_e = 1$.



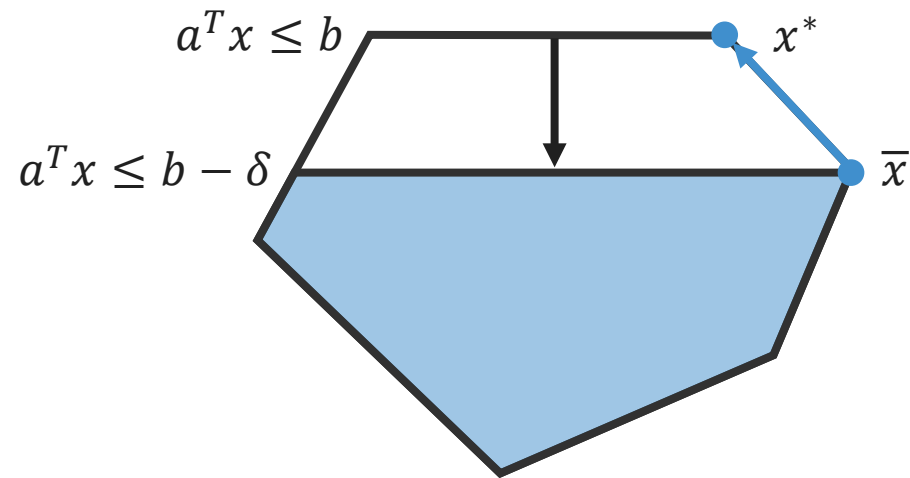
Edge Removal

Theorem. There is an $O(\text{max degree})$ -approximation algorithm for stabilizing by removing edges.

- Idea of the algorithm:
 - Use the capacity reduction algorithm.
 - For each vertex that would have its capacity reduced, remove all edges incident with it, except the ones with $x_e = 1$.
- Key property of our algorithm:
 - In unit-weight ($w = 1$) graphs, it completely preserves the total value that the players can get.

Polyhedral Tools

Theorem. Let $\mathcal{P}$ be any polytope, $a^T x \leq b$ be an inequality of the description of $\mathcal{P}$, and $\delta \in \mathbb{R}_{\geq 0}$. Let $\bar{x}$ be an optimal solution of the LP $\max\{c^T x : x \in \mathcal{P}, a^T x \leq b - \delta\}$, and assume that $\bar{x}$ is a non-optimal vertex of the LP $\max\{c^T x : x \in \mathcal{P}\}$. Furthermore, assume that there is no vertex $\tilde{x}$ of $\mathcal{P}$ satisfying $b - \delta < a^T \tilde{x} < b$. Then it is possible to move to an optimal solution x^* of $\max\{c^T x : x \in \mathcal{P}\}$ from $\bar{x}$ in one step over the edges of $\mathcal{P}$ (i.e., there is an optimal vertex of $\mathcal{P}$ adjacent to $\bar{x}$).



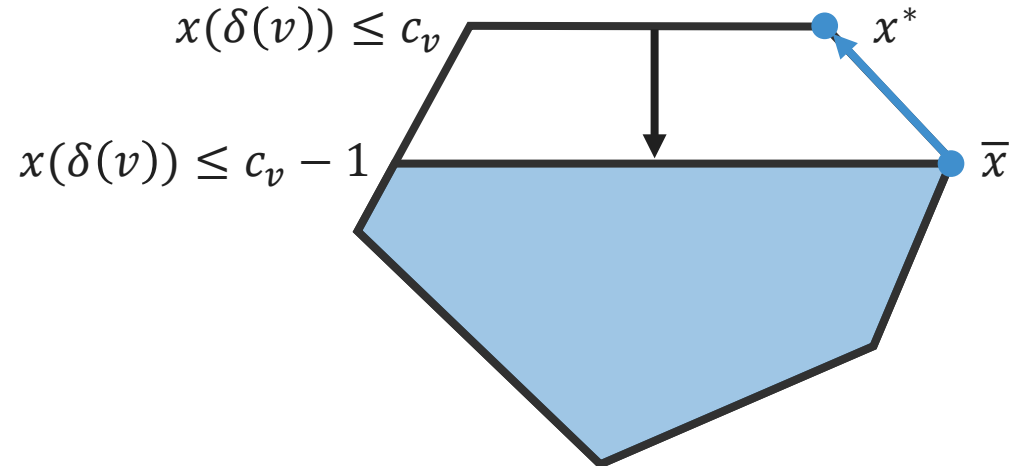
Polyhedral Tools

- Let's apply the theorem to the fractional c-matching polytope:

$$\mathcal{P} = \{x \in \mathbb{R}^E : x(\delta(v)) \leq c_v \ \forall v \in V, 0 \leq x \leq 1\}.$$

- In this polytope there is only a few ways this step can look, and all of them increase the number of odd cycles induced by the $x_e = \frac{1}{2}$ edges, by at most one.

$\Rightarrow$ The minimum number of odd cycles induced by the $x_e = \frac{1}{2}$ edges is a lower bound on the amount of capacity that has to be reduced.



Conclusion

- Stabilizing by reducing capacity is polynomial-time solvable.
- There is an $O(\text{max degree})$ -approximation algorithm for stabilizing by removing edges.
- We used (new) polyhedral techniques to prove our results.

Open problems

- Stabilizing by removing vertices when $c \leq 2$.
- Approximation algorithm for stabilizing by removing edges while simultaneously preserving the total value that players can get.

Thank you!

Questions?

References

- [AHS18] Ahmadian, Hosseinzadeh, Sanità. Stabilizing network bargaining games by blocking players. (2018)
- [BH+10] Bateni, Hajiaghayi, Immorlica, Mahini. The cooperative game theory foundations of network bargaining games. (2010)
- [BC+15] Bock, Chandrasekaran, Könemann, Peis, Sanità. Finding small stabilizers for unstable graphs. (2015)
- [IK+] Ito, Kakimura, Kamiyama, Kobayashi, Okamoto. Efficient stabilization of cooperative matching games. (2017)
- [KS20] Koh, Sanità. Stabilizing weighted graphs. (2020)